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A class of boundary value problems for first-order impulsive integro-differential equations with deviating arguments☆

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ABSTRACT

This paper is concerned with a class of boundary value problems for nonlinear mixed impulsive integro-differential equations with deviating arguments. We establish a new comparison principle and use the method of upper and lower solutions together with the monotone iterative technique. Under suitable conditions, we obtain the existence results of extremal solutions for the problems. An example is also given to illustrate our results.

1. Introduction

Impulsive differential equations have become increasingly important in recent years in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology, and economics. There has been a significant development in impulse theory. In particular, there is an increasing interest in the study of nonlinear mixed integro-differential equations with deviating arguments and multipoint boundary value problems (BVPs) [1–4] for impulsive differential equations.

In [5], the method of lower and upper solutions combined with the monotone iterative technique and the numerical-analytic method were applied to study the problem

$$\begin{cases} x'(t) = f\left(t, x(t), \displaystyle\int_0^T k(s)x(s)ds\right) & t \in J = [0, T] \\ x(0) = \lambda x(T) + \displaystyle\int_0^T D(s)x(s)ds + d & d \in R, \end{cases}$$

where $f \in C[J \times R^2, R]$, f is non-decreasing with respect to the third variable, $k, D \in C[J, R_+]$, and $\lambda \geq 0$.

Chen and Shen [6] studied

$$\begin{cases} u'(t) = f(t, u(t), u(\theta(t))) & t \neq t_k,\ t \in J = [0, T] \\ \Delta u(t_k) = I_k(u(t_k)) & k = 1, 2, \ldots, m \\ u(0) + \mu \displaystyle\int_0^T u(s)ds = u(T), \end{cases}$$

where $f \in C[J \times R^2, R]$, and $\mu = 1$ or -1, by the method of upper and lower solutions and the monotone iterative technique.

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Motivated by the above, we are concerned with the following BVPs of nonlinear mixed impulsive integro-differential equations with deviating arguments:

$$\begin{cases} u'(t) = f(t, u(t), u(\alpha(t)), Tu, Su) \quad t \neq t_k,\ t \in J = [0, T] \\ \Delta u(t_k) = I_k(u(t_k)) \quad k = 1, 2, \ldots, m \\ u(0) = \lambda_1 u(T) + \lambda_2 u(\eta) + \lambda_3 \displaystyle\int_0^T w(s, u(s))ds + k, \end{cases} \tag{1.1}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $I_k \in C(R, R)$, f is continuous everywhere except at $\{t_k\} \times R^4$; $f(t_k^+, \cdot, \cdot, \cdot, \cdot)$ and $f(t_k^-, \cdot, \cdot, \cdot, \cdot)$ exist with $f(t_k^-, \cdot, \cdot, \cdot, \cdot) = f(t_k, \cdot, \cdot, \cdot, \cdot)$;

$$(Tu)(t) = \int_0^{\beta(t)} k(t, s)u(\gamma(s))ds, \qquad (Su)(t) = \int_0^T h(t, s)u(\delta(s))ds,$$

and $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $w \in C(J \times R, R)$, $0 \leq \lambda_1 \leq 1$, $0 \leq \lambda_2$, $0 \leq \lambda_3$, $k \in R$, and $0 \leq \eta \leq T$. The assumptions concerning $\alpha, \beta, \gamma, \delta, k$, and h will be given latter. The boundary conditions in Eq. (1.1) involve several special cases such as periodic boundary conditions, anti-periodic boundary conditions, integral boundary, and initial problems.

Special cases

(i) If $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = k = 0$, then Eq. (1.1) reduces to the periodic boundary value problem (cf. [7–11]).
(ii) If $\lambda_2 = 1 + \lambda_1$, $\eta = 0$, and $\lambda_3 = k = 0$, then Eq. (1.1) reduces to the anti-periodic boundary value problem (cf. [12–15]).
(iii) If $\lambda_3 \neq 0$ and $\lambda_2 = k = 0$, then Eq. (1.1) reduces to the integral boundary value problems which have been studied in [16–19].
(iv) If $\lambda_1 = \lambda_2 = \lambda_3 = 0$, then Eq. (1.1) reduces to initial problems (cf. [20,21]).

For example, if $\lambda_2 = 1 + \lambda_1$, $\eta = 0$, and $\lambda_3 = k = 0$, Eq. (1.1) reduces to an anti-periodic boundary value problem, which was considered by Wang and Zhang in [15]. There, the existence results of quasi-extremal solutions for the anti-periodic boundary value problems was obtained by the method of upper and lower solutions with the monotone iterative technique. Therefore, we extend some previous results in many respects.

The article is organized as follow. In Section 2, we establish a new comparison principle. In Section 3, by using of the monotone iterative technique and the method of upper and lower solutions, we obtain the existence results of extremal solutions for (1.1). In Section 4, we give an example that illustrates our results.

2. Preliminaries and lemmas

Let $PC(J) = \{x : J \to R; x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k) = x(t_k^-),\ k = 1, 2, \ldots, m\}$; $PC^1(J) = \{x \in PC(J) : x'(t)$ is continuous everywhere except for some t_k at which $x'(t_k^+)$ and $x'(t_k^-)$ exist and $x'(t_k) = x'(t_k^-),\ k = 1, 2, \ldots, m\}$. Let $J^- = J \setminus \{t_k,\ k = 1, 2, \ldots, m\}$; $PC(J)$ and $PC^1(J)$ are Banach spaces with the norms $\|x\|_{PC} = \sup\{|x(t)| : t \in J\}$ and $\|x\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\}$. $x \in PC^1(J)$ is called a solution of BVPs (1.1) if it satisfies Eq. (1.1).

In what follows, we need the following hypotheses.

(H_1) $\alpha, \beta, \gamma, \delta \in C(J, J)$, $N, K, H \in C(J, R_+)$, $k \in C(\Omega, R_+)$, $h \in C(J^2, R_+)$, $R_+ = [0, +\infty)$, $\Omega = \{(t, s) \in J^2 \mid 0 \leq s \leq \beta(t)\}$, $M \in C(J, R)$, $\int_0^T M(\tau)d\tau \geq 0$, and $0 \leq L_k \leq 1$, $0 < r \leq 1$.
For convenience, we set

$$\begin{cases} N^*(t) = N(t)e^{\int_0^t M(s)ds}e^{-\int_0^{\alpha(t)} M(s)ds}, \qquad K^*(t) = K(t)e^{\int_0^t M(s)ds}, \\ H^*(t) = H(t)e^{\int_0^t M(s)ds}, \qquad k^*(t, s) = k(t, s)e^{-\int_0^{\gamma(s)} M(\tau)d\tau}, \\ h^*(t, s) = h(t, s)e^{-\int_0^{\delta(s)} M(\tau)d\tau}, \qquad r^* = re^{-\int_0^T M(s)ds}, \end{cases} \tag{2.1}$$

$\theta^*(t) = N^*(t) + K^*(t)\int_0^{\beta(t)} k^*(t, s)ds + H^*(t)\int_0^T h^*(t, s)ds \not\equiv 0$ for $t \in J$, $\mu^* = \int_0^T \theta^*(t)dt$.

(H_2)

$$\left[\mu^* + \sum_{k=1}^m L_k\right] \leq r^*.$$

Lemma 2.1. *Assume that (H_1) and (H_2) hold, and that $q \in PC^1(J)$ such that*

$$\begin{cases} q'(t) \leq -M(t)q(t) - (\mathscr{H}q)(t) \quad t \neq t_k,\ t \in J = [0, T] \\ \Delta q(t_k) \leq -L_k(q(t_k)) \quad k = 1, 2, \ldots, m \\ q(0) \leq rq(T), \end{cases} \tag{2.2}$$

where the operator $\mathscr{H}$ is defined as

$$(\mathscr{H}q)(t) = N(t)q(\alpha(t)) + K(t)\int_0^{\beta(t)} k(t,s)q(\gamma(s))ds + H(t)\int_0^T h(t,s)q(\delta(s))ds.$$

Then $q(t) \le 0$ for $t \in J$.

Proof. Let $p(t) = q(t)e^{\int_0^t M(s)ds}$. Obviously $p(t)$ and $q(t)$ have the same sign on J. In view of (2.2), we have

$$\begin{cases} p'(t) \le -(\mathscr{H}^*p)(t) & t \ne t_k,\ t \in J = [0,T] \\ \Delta p(t_k) \le -L_k(p(t_k)) & k = 1,2,\ldots,m \\ p(0) \le r^*p(T), \end{cases} \tag{2.3}$$

where $(\mathscr{H}^*p)(t) = N^*(t)p(\alpha(t)) + K^*(t)\int_0^{\beta(t)} k^*(t,s)p(\gamma(s))ds + H^*(t)\int_0^T h^*(t,s)p(\delta(s))ds$.

Next, we will show that $p(t) \le 0$.

Suppose, to the contrary, that $p(t) > 0$ for some $t \in J$.

(i) If $p(t) \ge 0$, $p(t) \not\equiv 0$ for $t \in J$, we get $p'(t) \le 0$, in view of the first inequality of (2.3). By the second inequality in (2.3), we obtain that $p(t)$ is non-increasing in J. Then $0 \le p(T) \le p(t) \le p(0)$. On the other hand, by the third inequality in (2.3), if $r^* = 1$, then $p(T) \le p(t) \le p(0) \le p(T)$, so we get that $p(t) \equiv C > 0$. Hence $p'(t) \equiv 0$. By the first inequality in (2.3) again, we have

$$0 \le -C\theta^*(t) \quad \forall t \in J.$$

By (H_1), we get that $C \le 0$, which is a contradiction.

If $0 < r^* < 1$, then $p(T) \le p(0) \le r^*p(T)$, so $p(T)(1-r^*) \le 0$. We have $0 \le p(T) \le 0$. Since p is non-increasing in J, we infer that $p(t) \equiv 0$. This is a contradiction.

(ii) If $p(t^*) = \sup_{t\in J} p(t) > 0$, $p(t_*) = \inf_{t\in J} p(t) = -\lambda < 0$, then $\lambda > 0$.

Case 1. If $t_* < t^*$, integrating from t_* to t^*, we get, from (2.3), that

$$\begin{aligned} 0 < p(t^*) &= p(t_*) + \int_{t_*}^{t^*} p'(s)ds + \sum_{t_* \le t_k < t^*} \Delta p(t_k) \\ &\le -\lambda + \int_{t_*}^{t^*} -(\mathscr{H}^*p)(s)ds - \sum_{t_* \le t_k < t^*} L_k p(t_k) \\ &\le -\lambda + \mu^*\lambda + \lambda\sum_{k=1}^m L_k. \end{aligned}$$

Hence

$$1 < \mu^* + \sum_{k=1}^m L_k,$$

which is in contradiction to (H_2).

Case 2. If $t^* < t_*$, we have

$$\begin{aligned} 0 < p(t^*) &= p(0) + \int_0^{t^*} p'(s)ds + \sum_{0 < t_k < t^*} \Delta p(t_k) \\ &\le p(0) + \int_0^{t^*} -(\mathscr{H}^*p)(s)ds + \lambda\sum_{0 < t_k < t^*} L_k \\ &\le p(0) + \lambda\int_0^{t^*} \theta^*(s)ds + \lambda\sum_{0 < t_k < t^*} L_k, \end{aligned}$$

$$\begin{aligned} p(T) &= p(t_*) + \int_{t_*}^{T} p'(s)ds + \sum_{t_* \le t_k < T} \Delta p(t_k) \\ &\le -\lambda + \int_{t_*}^{T} -(\mathscr{H}^*p)(s)ds + \lambda\sum_{t_* \le t_k < T} L_k \\ &\le -\lambda + \lambda\int_{t_*}^{T} \theta^*(s)ds + \lambda\sum_{t_* \le t_k < T} L_k. \end{aligned}$$

By the two inequalities above, we obtain

$$\begin{aligned}-\lambda+\frac{1}{r^*}\lambda\int_{t_*}^{T}\theta^*(s)ds+\frac{1}{r^*}\lambda\sum_{t_*\le t_k<T}L_k &\ge -\lambda+\lambda\int_{t_*}^{T}\theta^*(s)ds+\lambda\sum_{t_*\le t_k<T}L_k\\ &\ge p(T)\ge\frac{1}{r^*}p(0)\\ &> -\frac{1}{r^*}\lambda\int_0^{t^*}\theta^*(s)ds-\frac{1}{r^*}\lambda\sum_{0<t_k<t^*}L_k\\ &\ge -\frac{1}{r^*}\lambda\int_0^{t_*}\theta^*(s)ds-\frac{1}{r^*}\lambda\sum_{0<t_k<t_*}L_k.\end{aligned}$$

Therefore, we get that $(\mu^*+\sum_{k=1}^{m}L_k)>r^*$, which is in contradiction to (H_2). Hence $p(t)\le 0$, $q(t)\le 0$. We complete the proof. □

Let $C_k, d\in R$, $\sigma\in PC(J)$. In order to deal with the following linear problem,

$$\begin{cases}u'(t)=-M(t)u(t)-(\mathscr{H}u)(t)+\sigma(t) & t\neq t_k,\ t\in J=[0,T]\\ \Delta u(t_k)=-L_k(u(t_k))+C_k & k=1,2,\ldots,m\\ u(0)=ru(T)+d,\end{cases}\tag{2.4}$$

we also need the following hypothesis:

$$(H_3)\ \varpi\equiv e^{\int_0^T|M(\tau)|d\tau}\left(1+\frac{r}{e^{\int_0^T M(\tau)d\tau}-r}\right)\left(\mu+\sum_{k=1}^{m}L_k\right)<1,\tag{2.5}$$

where $\mu=\int_0^T[N(t)+K(t)\int_0^{\beta(t)}k(t,s)ds+H(t)\int_0^T h(t,s)ds]dt$.

Lemma 2.2. *Suppose that (H_3) holds and that $0\le L_k\le 1$, $M\in C(J,R)$, $\int_0^T M(s)ds\ge 0$, $0<r\le 1$, and $\int_0^T M(s)ds>0$ if $r=1$. Then the following integral equation,*

$$\begin{aligned}u(t)=&\frac{de^{\int_t^T M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r}+\int_0^T G(t,s)(\sigma(s)-(\mathscr{H}u)(s))ds+\frac{re^{-\int_0^t M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r}\sum_{k=1}^{m}e^{\int_0^{t_k}M(\tau)d\tau}(-L_k(u(t_k))+C_k)\\ &+\sum_{0<t_k<t}e^{-\int_0^t M(\tau)d\tau}e^{\int_0^{t_k}M(\tau)d\tau}(-L_k(u(t_k))+C_k),\end{aligned}\tag{2.6}$$

where

$$G(t,s)=\begin{cases}\dfrac{e^{\int_t^T M(\tau)d\tau}e^{\int_0^s M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r}, & 0\le s\le t\le T,\\ \dfrac{re^{\int_t^s M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r}, & 0\le t\le s\le T,\end{cases}$$

has a unique solution u in $PC(J)$.

Proof. Define operator F by

$$\begin{aligned}(Fu)(t)=&\frac{de^{\int_t^T M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r}+\int_0^T G(t,s)(\sigma(s)-(\mathscr{H}u)(s))ds\\ &+\frac{re^{-\int_0^t M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r}\sum_{k=1}^{m}e^{\int_0^{t_k}M(\tau)d\tau}(-L_k(u(t_k))+C_k)+\sum_{0<t_k<t}e^{-\int_0^t M(\tau)d\tau}e^{\int_0^{t_k}M(\tau)d\tau}(-L_k(u(t_k))+C_k).\end{aligned}$$

If $0\le s\le t\le T$,

$$\begin{aligned}\frac{e^{\int_t^T M(\tau)d\tau}e^{\int_0^s M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r}&\le\frac{e^{\int_0^T M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r}\\ &=\frac{e^{\int_0^T M(\tau)d\tau}-r+r}{e^{\int_0^T M(\tau)d\tau}-r}\\ &=1+\frac{r}{e^{\int_0^T M(\tau)d\tau}-r},\end{aligned}$$

and if $0 \le t \le s \le T$,

$$\begin{aligned}
\frac{re^{\int_t^s M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r} &\le \frac{re^{\int_0^T M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r}\\
&\le \frac{e^{\int_0^T M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r}\\
&= 1+\frac{r}{e^{\int_0^T M(\tau)d\tau}-r},
\end{aligned}$$

so it is easy to see that

$$\max\{G(t,s),\ (t,s)\in J^2\} = 1+\frac{r}{e^{\int_0^T M(\tau)d\tau}-r}.$$

For any $x, y \in PC(J)$, we get

$$\begin{aligned}
\|(Fx)(t)-(Fy)(t)\|_{PC} &\le \left(1+\frac{r}{e^{\int_0^T M(\tau)d\tau}-r}\right)\int_0^T \mid (-\mathscr{H}x(s)+\mathscr{H}y(s))ds \mid\\
&\quad + \max\left|\frac{re^{-\int_0^t M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r}\sum_{k=1}^m e^{\int_0^{t_k} M(\tau)d\tau}(-L_k(x(t_k))+L_k(y(t_k)))\right.\\
&\quad \left. + \sum_{0<t_k<t} e^{-\int_0^t M(\tau)d\tau}e^{\int_0^{t_k} M(\tau)d\tau}(-L_k(x(t_k))+L_k(y(t_k)))\right|\\
&\le \left(1+\frac{r}{e^{\int_0^T M(\tau)d\tau}-r}\right)\int_0^T \mid (-\mathscr{H}x(s)+\mathscr{H}y(s))ds \mid\\
&\quad + \max\left\{\left(1+\frac{r}{e^{\int_0^T M(\tau)d\tau}-r}\right)\sum_{0<t_k<t} e^{-\int_{t_k}^t M(\tau)d\tau} \mid (-L_k(x(t_k))+L_k(y(t_k))) \mid\right.\\
&\quad \left. + \frac{r}{e^{\int_0^T M(\tau)d\tau}-r}\sum_{t\le t_k<T} e^{\int_t^{t_k} M(\tau)d\tau} \mid (-L_k(x(t_k))+L_k(y(t_k))) \mid\right\}\\
&\le \left(1+\frac{r}{e^{\int_0^T M(\tau)d\tau}-r}\right)\int_0^T \mid (-\mathscr{H}x(s)+\mathscr{H}y(s))ds \mid\\
&\quad + \left(1+\frac{r}{e^{\int_0^T M(\tau)d\tau}-r}\right)\max\left\{\sum_{0<t_k<t} e^{-\int_{t_k}^t M(\tau)d\tau} \mid (-L_k(x(t_k))+L_k(y(t_k))) \mid\right.\\
&\quad \left. + \sum_{t\le t_k<T} e^{\int_t^{t_k} M(\tau)d\tau} \mid (-L_k(x(t_k))+L_k(y(t_k))) \mid\right\}\\
&\le \left(1+\frac{r}{e^{\int_0^T M(\tau)d\tau}-r}\right)\int_0^T \mid (-\mathscr{H}x(s)+\mathscr{H}y(s))ds \mid\\
&\quad + \left(1+\frac{r}{e^{\int_0^T M(\tau)d\tau}-r}\right)\max\left\{\sum_{0<t_k<t} e^{\int_0^T |M(\tau)|d\tau} \mid (-L_k(x(t_k))+L_k(y(t_k))) \mid\right.\\
&\quad \left. + \sum_{t\le t_k<T} e^{\int_0^T |M(\tau)|d\tau} \mid (-L_k(x(t_k))+L_k(y(t_k))) \mid\right\}\\
&\le e^{\int_0^T |M(\tau)|d\tau}\left(1+\frac{r}{e^{\int_0^T M(\tau)d\tau}-r}\right)\left(\mu+\sum_{k=1}^m L_k\right)\|x-y\|_{PC}\\
&= \varpi\|x-y\|_{PC},
\end{aligned}$$

which implies from the Banach fixed point theorem that F has a unique fixed point u in $PC(J)$. The proof is complete. □

Lemma 2.3. *Assume that (H_1)–(H_3) hold and that $\int_0^T M(s)ds > 0$ if $r = 1$. Then the linear problem (2.4) has a unique solution $u \in PC^1(J, E)$, and it is represented by the integral equation (2.6).*

Proof. By Lemma 2.2, the integral equation (2.6) has a unique solution $u \in PC(J)$. Differentiating (2.6), we obtain

$$\begin{aligned}
u'(t) &= \frac{d}{dt}\left[\frac{de^{\int_t^T M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r} + \int_0^T G(t,s)(\sigma(s) - (\mathscr{H}u)(s))ds + \frac{re^{-\int_0^t M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r}\right.\\
&\quad \left.\times \sum_{k=1}^m e^{\int_0^{t_k} M(\tau)d\tau}(-L_k(u(t_k)) + C_k) + \sum_{0<t_k<t} e^{-\int_0^t M(\tau)d\tau} e^{\int_0^{t_k} M(\tau)d\tau}(-L_k(u(t_k)) + C_k)\right]\\
&= -M(t)\left[\frac{de^{\int_t^T M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r} + \int_0^T G(t,s)(\sigma(s) - (\mathscr{H}u)(s))ds + \frac{re^{-\int_0^t M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r}\right.\\
&\quad \left.\times \sum_{k=1}^m e^{\int_0^{t_k} M(\tau)d\tau}(-L_k(u(t_k)) + C_k) + \sum_{0<t_k<t} e^{-\int_0^t M(\tau)d\tau} e^{\int_0^{t_k} M(\tau)d\tau}(-L_k(u(t_k)) + C_k)\right]\\
&\quad + \left(\frac{-r}{e^{\int_0^T M(\tau)d\tau}-r} + \frac{e^{\int_0^T M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r}\right)(\sigma(t) - (\mathscr{H}u)(t))\\
&= -M(t)u(t) - (\mathscr{H}u)(t) + \sigma(t) \quad t \in J^-,\\
\Delta u(t_k) &= u(t_k^+) - u(t_k^-)\\
&= \sum_{0<t_j\le t_k} \Delta u(t_j) - \sum_{0<t_j<t_k} \Delta u(t_j)\\
&= \sum_{j=1}^{k}(-L_j(u(t_j)) + C_j) - \sum_{j=1}^{k-1}(-L_j(u(t_j)) + C_j)\\
&= -L_k(u(t_k)) + C_k.
\end{aligned}$$

Also,

$$\begin{aligned}
u(0) &= \frac{r}{e^{\int_0^T M(\tau)d\tau}-r}\sum_{k=1}^m e^{\int_0^{t_k} M(\tau)d\tau}(-L_k(u(t_k)) + C_k)\\
&\quad + \int_0^T \frac{re^{\int_0^s M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r}(\sigma(s) - (\mathscr{H}u)(s))ds + \frac{de^{\int_0^T M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r},\\
u(T) &= \frac{1}{e^{\int_0^T M(\tau)d\tau}-r}\sum_{k=1}^m e^{\int_0^{t_k} M(\tau)d\tau}(-L_k(u(t_k)) + C_k)\\
&\quad + \int_0^T \frac{e^{\int_0^s M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r}(\sigma(s) - (\mathscr{H}u)(s))ds + \frac{d}{e^{\int_0^T M(\tau)d\tau}-r}.
\end{aligned}$$

It is easy to check that $u(0) = ru(T) + d$.

Hence, we infer that u defined by (2.6) is a solution of (2.4).

Next, we show that problem (2.4) is uniquely solvable. Let u_1, u_2 be two solutions of (2.4) and set $p = u_1 - u_2$. We have

$$\begin{aligned}
p' &= u_1' - u_2'\\
&= -M(t)u_1(t) - (\mathscr{H}u_1)(t) + \sigma(t) - (-M(t)u_2(t) - (\mathscr{H}u_2)(t) + \sigma(t))\\
&= -M(t)p(t) - (\mathscr{H}p)(t) \quad t \in J^-,\\
\Delta p(t_k) &= \Delta u_1 - \Delta u_2\\
&= -L_k u_1(t_k) + C_k - (-L_k u_2(t_k) + C_k)\\
&= -L_k p(t_k),\\
p(0) &= u_1(0) - u_2(0)\\
&= ru_1(T) + d - (ru_2(T) + d)\\
&= rp(T).
\end{aligned}$$

In view of Lemma 2.1, we get that $p \le 0$, which implies that $u_1 \le u_2$. Similarly, we can get that $u_1 \ge u_2$. Hence $u_1 = u_2$. The proof is complete. □

3. Main results

If $u_0, v_0 \in PC^1(J)$ and $u_0(t) \le v_0(t), \ \forall t \in J$, we define the interval

$$[u_0, v_0] = \{x \in PC^1(J) : u_0(t) \le x(t) \le v_0(t),\ t \in J\}.$$

In what follows, we also need the following assumptions.

(H_4) There exist $u_0, v_0 \in PC^1(J)$ with $u_0(t) \le v_0(t),\ \forall t \in J$ such that

$$\begin{cases} u_0'(t) \le f(t, u_0(t), u_0(\alpha(t)), Tu_0, Su_0) \quad t \ne t_k,\ t \in J = [0, T] \\ \Delta u_0(t_k) \le I_k(u_0(t_k)) \quad k = 1, 2, \dots, m \\ u_0(0) \le \lambda_1 u_0(T) + \lambda_2 u_0(\eta) + \lambda_3 \displaystyle\int_0^T w(s, u_0(s))ds + k, \\ v_0'(t) \ge f(t, v_0(t), v_0(\alpha(t)), Tv_0, Sv_0) \quad t \ne t_k,\ t \in J = [0, T] \\ \Delta v_0(t_k) \ge I_k(v_0(t_k)) \quad k = 1, 2, \dots, m \\ v_0(0) \ge \lambda_1 v_0(T) + \lambda_2 v_0(\eta) + \lambda_3 \displaystyle\int_0^T w(s, v_0(s))ds + k. \end{cases} \tag{3.1}$$

(H_5)

$$\begin{aligned} f(t, \bar{u}, \bar{u}(\alpha(t)), T\bar{u}, S\bar{u}) - f(t, u, u(\alpha(t)), Tu, Su) \ \ge\ & -M(t)(\bar{u} - u) - N(t)(\bar{u} - u)(\alpha(t)) \\ & - K(t)T(\bar{u} - u) - H(t)S(\bar{u} - u), \end{aligned} \tag{3.2}$$

$$I_k(\bar{u}) - I_k(u) \ge -L_k(\bar{u} - u), \tag{3.3}$$

for all $u_0(t) \le u(t) \le \bar{u}(t) \le v_0(t)$ in J.

(H_6) Assume that $a(t)$ is a non-negative integrable function, such that

$$w(t, \bar{u}) - w(t, u) \ge a(t)(\bar{u} - u), \tag{3.4}$$

for all $u_0(t) \le u(t) \le \bar{u}(t) \le v_0(t)$ in J.

Theorem 3.1. *Assume that (H_1)–(H_6) hold and that $\int_0^T M(s)ds > 0$ if $r = 1$. Then there exist two monotone sequences $\{u_n(t)\}, \{v_n(t)\} \subset PC^1(J)$ with*

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots \le v_n \le \cdots \le v_1 \le v_0, \tag{3.5}$$

such that $\lim_{n\to\infty} u_n = u^*(t)$, $\lim_{n\to\infty} v_n = v^*(t)$ uniformly on J. Moreover, $u^*(t)$ and $v^*(t)$ are the minimal solution and the maximal solution of (1.1) in $[u_0, v_0]$, respectively.

Proof. For $\xi \in [u_0, v_0]$, we consider (2.4) with

$$\sigma(t) = M(t)\xi(t) + (\mathscr{H}\xi)(t) + f(t, \xi(t), \xi(\alpha(t)), T\xi, S\xi), \qquad C_k = I_k(\xi(t_k)) + L_k\xi(t_k),$$
$$d = \lambda_2\xi(\eta) + \lambda_3 \int_0^T w(s, \xi(s))ds + k, \quad r = \lambda_1.$$

By Lemma 2.3, the BVP (2.4) has a unique solution $u \in PC(J)$.

We define an operator $A : [u_0, v_0] \to PC(J)$ by $u = A\xi$. We claim that

(a) $u_0 \le Au_0, Av_0 \le v_0$,
(b) A is nondecreasing on $[u_0, v_0]$.

We first prove (a). Set $u_1 = Au_0$, $p(t) = u_0(t) - u_1(t)$. Then we have

$$\begin{aligned} p' &= u_0' - u_1' \\ &\le f(t, u_0(t), u_0(\alpha(t)), Tu_0, Su_0) - [f(t, u_0(t), u_0(\alpha(t)), Tu_0, Su_0) \\ &\quad + M(t)u_0(t) + (\mathscr{H}u_0)(t) - M(t)u_1(t) - (\mathscr{H}u_1)(t)] \\ &= -M(t)p(t) - (\mathscr{H}p)(t) \quad t \in J^-, \end{aligned}$$

$$\begin{aligned} \Delta p(t_k) &= \Delta u_0(t_k) - \Delta u_1(t_k) \\ &\le I_k(u_0(t_k)) - [I_k(u_0(t_k)) - L_k(u_1 - u_0)] \\ &= -L_k p(t_k), \end{aligned}$$

$$\begin{aligned} p(0) &= u_0(0) - u_1(0) \\ &\le \lambda_1 u_0(T) + \lambda_2 u_0(\eta) + \lambda_3 \int_0^T w(s, u_0(s))ds + k - (\lambda_1 u_1(T) + \lambda_2 u_0(\eta) + \lambda_3 \int_0^T w(s, u_0(s))ds + k) \\ &= \lambda_1 p(T). \end{aligned}$$

By Lemma 2.1, we have $p \le 0$. That is, $u_0 \le Au_0$. Similarly, we can prove that $Av_0 \le v_0$.

To prove (b), set $\gamma_1, \gamma_2 \in [u_0, v_0]$ and $\gamma_1 \le \gamma_2$, $\gamma_1^* = A\gamma_1$, $\gamma_2^* = A\gamma_2$, $p = \gamma_1^* - \gamma_2^*$. Then

$$\begin{aligned}
p'(t) &= \gamma_1'^* - \gamma_2'^* \\
&= f(t, \gamma_1(t), \gamma_1(\alpha(t)), T\gamma_1, S\gamma_1) + M(t)\gamma_1(t) + (\mathscr{H}\gamma_1)(t) - M(t)\gamma_1^*(t) - (\mathscr{H}\gamma_1^*)(t) \\
&\quad - [f(t, \gamma_2(t), \gamma_2(\alpha(t)), T\gamma_2, S\gamma_2) + M(t)\gamma_2(t) + (\mathscr{H}\gamma_2)(t) - M(t)\gamma_2^*(t) - (\mathscr{H}\gamma_2^*)(t)] \\
&\le -M(t)p(t) - (\mathscr{H}p)(t) \quad t \in J^-, \\
\Delta p(t_k) &= \Delta\gamma_1^*(t_k) - \Delta\gamma_2^*(t_k) \\
&= I_k(\gamma_1(t_k)) - L_k(\gamma_1^*(t_k) - \gamma_1(t_k)) - (I_k(\gamma_2(t_k)) - L_k(\gamma_2^*(t_k) - \gamma_2(t_k))) \\
&= I_k(\gamma_1(t_k)) - I_k(\gamma_2(t_k)) + L_k(\gamma_1 - \gamma_2) - L_k(\gamma_1^* - \gamma_2^*) \\
&\le -L_k p(t_k), \\
p(0) &= \gamma_1^*(0) - \gamma_2^*(0) \\
&\le \lambda_1\gamma_1^*(T) + \lambda_2\gamma_1(\eta) + \lambda_3\int_0^T w(s, \gamma_1(s))ds + k - (\lambda_1\gamma_2^*(T) + \lambda_2\gamma_2(\eta) + \lambda_3\int_0^T w(s, \gamma_2(s))ds + k) \\
&\le \lambda_1 p(T) + \lambda_2(\gamma_1(\eta) - \gamma_2(\eta)) + \lambda_3\int_0^T a(s)(\gamma_1(s) - \gamma_2(s))ds \\
&\le \lambda_1 p(T).
\end{aligned}$$

In view of Lemma 2.1, we get that $A\gamma_1 \le A\gamma_2$. Hence (b) holds.

Now, we define two sequences $\{u_n\}$, $\{v_n\}$ in $PC^1(J)$:

$$u_{n+1} = Au_n, \qquad v_{n+1} = Av_n \quad (n = 0, 1, 2, \ldots).$$

By (a) and (b), we have that (3.5) holds. And, for any $n = 1, 2, \ldots$, u_n and $v_n \in PC^1(J)$ satisfy the following integro-differential equations:

$$\begin{cases}
u_n'(t) = f(t, u_{n-1}(t), u_{n-1}(\alpha(t)), Tu_{n-1}, Su_{n-1}) - M(t)(u_n(t) - u_{n-1}(t)) \\
-(\mathscr{H}(u_n - u_{n-1}))(t) \quad t \ne t_k,\ t \in J = [0, T] \\
\Delta u_n(t_k) = -L_k u_n(t_k) + I_k(u_{n-1}(t_k)) + L_k u_{n-1}(t_k) \quad k = 1, 2, \ldots, m \\
u_n(0) = \lambda_1 u_n(T) + \lambda_2 u_{n-1}(\eta) + \lambda_3\int_0^T w(s, u_{n-1}(s))ds + k,
\end{cases}$$

$$\begin{cases}
v_n'(t) = f(t, v_{n-1}(t), v_{n-1}(\alpha(t)), Tv_{n-1}, Sv_{n-1}) - M(t)(v_n - v_{n-1}) \\
-(\mathscr{H}(v_n - v_{n-1}))(t) \quad t \ne t_k,\ t \in J = [0, T] \\
\Delta v_n(t_k) = -L_k v_n(t_k) + I_k(v_{n-1}(t_k)) + L_k v_{n-1}(t_k) \quad k = 1, 2, \ldots, m \\
v_n(0) = \lambda_1 v_n(T) + \lambda_2 v_{n-1}(\eta) + \lambda_3\int_0^T w(s, v_{n-1}(s))ds + k.
\end{cases}$$

Therefore, we have that $\{u_n\}$, $\{v_n\}$ are monotonically and uniformly convergent to $u^*(t)$ and $v^*(t)$ on J, respectively. It is not difficult to prove that $u^*(t)$, $v^*(t)$ are solutions of Eq. (1.1).

Finally, we assert that, if $u \in [u_0, v_0]$ is any solution of (1.1), then $u^*(t) \le u(t) \le v^*(t)$ on J. We will prove that, if $u_n \le u \le v_n$, for $n = 0, 1, 2, \ldots$, then $u_{n+1}(t) \le u(t) \le v_{n+1}(t)$.

Set $p(t) = u_{n+1}(t) - u(t)$. Then

$$\begin{aligned}
p'(t) &= u_{n+1}' - u'(t) \\
&= f(t, u_n(t), Tu_n, Su_n) + M(t)u_n(t) + (\mathscr{H}u_n)(t) \\
&\quad - M(t)u_{n+1}(t) - (\mathscr{H}u_{n+1})(t) - f(t, u(t), u(\alpha(t)), Tu, Su) \\
&\le -M(t)(u_{n+1}(t) - u(t)) - (\mathscr{H}(u_{n+1} - u))(t) \\
&\le -M(t)p(t) - (\mathscr{H}p)(t) \quad t \in J^-, \\
\Delta p(t_k) &= \Delta u_{n+1}(t_k) - \Delta u(t_k) \\
&= I_k(u_n(t_k) - L_k(u_{n+1}(t_k) - u_n(t_k)) - I_k u(t_k)) \\
&\le -L_k(u_n(t_k) - u(t_k)) - L_k(u_{n+1}(t_k) - u_n(t_k)) \\
&= -L_k(u_{n+1}(t_k) - u(t_k)) \\
&= -L_k p(t_k), \\
p(0) &= u_{n+1}(0) - u(0) \\
&\le \lambda_1 u_{n+1}(T) + \lambda_2 u_n(\eta) + \lambda_3\int_0^T w(s, u_n(s))ds + k - (\lambda_1 u(T) + \lambda_2 u(\eta) + \lambda_3\int_0^T w(s, u(s))ds + k) \\
&\le \lambda_1 p(T) + \lambda_2(u_n(\eta) - u(\eta)) + \lambda_3\int_0^T a(s)(u_n(s) - u(s))ds \\
&\le \lambda_1 p(T).
\end{aligned}$$

By Lemma 2.1, we have that $p(t) \le 0$ for all $t \in J$. That is, $u_{n+1}(t) \le u(t)$. Similarly, we can prove that $u(t) \le v_{n+1}(t)$ for all $t \in J$. Thus $u_{n+1}(t) \le u(t) \le v_{n+1}(t)$ for all $t \in J$, which implies that $u^*(t) \le u(t) \le v^*(t)$. The proof is complete. $\square$

4. Example

Consider the following problems:

$$\begin{cases} u'(t) = \dfrac{t^4 u(t)}{100} - \dfrac{t}{600}\sin\left(u\left(\dfrac{t}{2}\right)\right) - \dfrac{t}{100}\displaystyle\int_0^t su(s)ds - \dfrac{t^3}{1000}\displaystyle\int_0^1 u(s)ds \quad t \neq \dfrac{1}{2},\ t \in J = [0,1] \\ \Delta u\left(\dfrac{1}{2}\right) = -\dfrac{27}{160}u^3\left(\dfrac{1}{2}\right) \\ u(0) = \dfrac{1}{2}u(1) + \dfrac{1}{100}u(\eta) + \dfrac{1}{100}\displaystyle\int_0^1 (u(s)-s)ds + \dfrac{1}{150} \quad \eta \in [0,1]. \end{cases} \tag{4.1}$$

Let $f(t,x,y,z,w) = \frac{t^4 x}{100} - \frac{t}{600}y - \frac{1}{100}z - t^3 w$, $M(t) = 0$, $N(t) = \frac{t}{600}$, $K(t) = \frac{1}{100}$, $H(t) = t^3$, $k(t,s) = ts$, $h(t,s) = \frac{1}{1000}$, $Tu(t) = t\int_0^t su(s)ds$, $Su(t) = \int_0^1 \frac{1}{1000}u(s)ds$, $\alpha(t) = \frac{t}{2}$, $\beta(t) = t$, $\gamma(s) = s$, $\delta(s) = s$, $w(s,u(s)) = u(s) - s$.

We can easily verify that (4.1) admits the lower solution $u_0(t) = 0$ and the upper solution

$$v_0(t) = \begin{cases} \dfrac{2}{3}t + 1, & t \in \left[0, \dfrac{1}{2}\right], \\ \dfrac{2}{3}t + \dfrac{2}{3}, & t \in \left(\dfrac{1}{2}, 1\right], \end{cases}$$

and that $u_0(t) \le v_0(t)$. It is easy to see that

$$\begin{aligned} I_k(x(t_k)) - I_k(y(t_k)) &= -\frac{27}{160}(x^3(t_k) - y^3(t_k)) \\ &\ge -\frac{3}{10}(x(t_k) - y(t_k)) \\ &= -L_1(x(t_k) - y(t_k)), \end{aligned}$$

where $u_0(t_k) \le y(t_k) \le x(t_k) \le v_0(t_k)$, $L_1 = \frac{3}{10}$.

Obviously,

$$\begin{aligned} f(t,\bar{u},\bar{u}(\alpha(t)),T\bar{u},S\bar{u}) - f(t,u,u(\alpha(t)),Tu,Su) \ \ge\ & -M(t)(\bar{u}-u) - N(t)(\bar{u}-u)(\alpha(t)) \\ & - K(t)T(\bar{u}-u) - H(t)S(\bar{u}-u), \end{aligned}$$

$$W(t,\bar{u}(t)) - W(t,u(t)) = \bar{u}(t) - u(t) \ge \frac{t}{3}(\bar{u}(t) - u(t)),$$

for all $u_0(t) \le u(t) \le \bar{u}(t) \le v_0(t)$ in J.

And we can check that $r^* = r = \frac{1}{2}$, $[\mu^* + \sum_{k=1}^m L_k] < r^*$, $e^{\int_0^T |M(\tau)|d\tau}(1 + \frac{r}{e^{\int_0^T M(\tau)d\tau} - r})(\mu + \sum_{k=1}^m L_k) < 0.91 < 1$. Then all conditions of Theorem 3.1 are satisfied. Therefore, the conclusion of Theorem 3.1 holds for problem (4.1).

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